Brain Teaser

By Rex Macey, CIMA®, CFA®

Question: You live in the land of RORO (risk-on, risk-off). Your only two investments are cash and one stock. Cash earns nothing. Each day the stock either doubles or halves and you may trade once a day. The up and down moves of the stock are completely random. Can you invest so as to expect a profit? If so, how? If not, why not? Assume 250 trading days and a long-only constraint.

Answer: The inspiration for this puzzle comes from the book, *Fortune's Formula: The Untold Story of the Scientific Betting System that Beat the Casinos* and *Wall Street* by William Poundstone. The interesting part of this teaser is that stocks and cash both have a zero geometric average rate of return, so it's not obvious there's a way to make money. Further, the high volatility of the stock suggests (incorrectly) that a mean-variance optimizer might prefer cash. In fact, the stock has an arithmetic return of ½ × (100%) + ½ × (–50%) = 25%.

There is a way to make money here. Split your money evenly between cash and the stock and rebalance every day. Let's look at a simple example of two days.

Start on the first day with $50 in cash and $50 in stock. The stock moves up so the portfolio is worth $150. Rebalance to $75 in cash and $75 in stock. The second day the stock drops, so the portfolio is worth $75 in cash + $37.50 in stock = $112.50. You've still made money. The order doesn't matter. If the stock dropped on the first day, you'd have $50 in cash + $25 in stock = $75. After rebalancing, you'd have $37.50 in each. When the stock doubled the second day, you'd have $112.50.

Obviously, making money is not guaranteed. Stocks could drop every day and you'd be better off with cash in that event. But you can expect to make money. To show how powerful this "diversify and rebalance" strategy can be, I ran 3,000 random simulations in Excel (spreadsheet available upon request) starting with $100. The simulations were based on 250 (days) × 3,000 (trials) = 750,000 random numbers generated with values greater than or equal to 0 and less than 1. If the random number is less than 0.5, it's a down day for the stock and it loses 50 percent of its value, otherwise it's an up day and the stock gains 100 percent. I created 3,000 ending values for both the 100-percent stock portfolio and the rebalancing strategy described above (the ending value of cash-only would always be $100).

Table 1 shows the distribution of results for each set of 3,000 simulations.

Notice that in all but four of the 3,000 trials, the strategy produced a terminal value greater than $100—in other words, a profit. The averages are large because getting a long lucky streak in the stock significantly increases the

TABLE 1: RESULTS OF 3,000 RANDOM SIMULATIONS

Percentile Rank	100% Stock	Rebalancing Strategy
0%		$7
25%	$0.02	$3,871,554
50%	$100	$247,779,462
75%	$409,600	$15,857,885,579
100%	$112,589,990,684,262,000	$8,314,099,114,225,150
Average	$136,782,360,616,049	$16,413,418,118,308
#>$100	1,425	2,996

TABLE 2: MARTINGALE—STOP AFTER 1ST WIN OR BANKRUPTCY

Init Stock	$40	$50	$60	$70	$80	$90	$100
Stop	1st Win	1st Win	1st Win	1st Win	1st Win	1st Win	1st Win
Percentile Rank							
0%	$1	$0	$50	$150	$100	$50	$0
25%	$100	$100	$90	$155	$120	$85	$100
50%	$20	$125	$160	$170	$140	$190	$200
75%	$140	$150	$160	$170	$180	$190	$200
100%	$140	$150	$160	$170	$180	$190	$200
Average	$119	$121	$132	$163	$153	$144	$135
#>$100	2,229	2,229	2,245	3,000	3,000	2,237	1,536

TABLE 3: MARTINGALE—STOP WHEN OUT OF TIME							
Init Stock	**$40**	**$50**	**$60**	**$70**	**$80**	**$90**	**$100**
Stop	**Out of Time**	**Out of Time**	**Out of Time**	**Out of Time**	**Out of Time**	**Out of Time**	**Out of Time**
Percentile Rank							
0%	$0	$0	$30	$90	$60	$31	$0
25%	$1,023	$98	$3,062	$5,236	$3,417	$1,633	$0
50%	$66,639	$19,077	$140,949	$214,639	$157,857	$77,948	$100
75%	$20,971,520	$6,553,600	$28,626,913	$49,313,602	$36,111,532	$14,439,124	$102,400
100%	4.06E + 20	6.46E + 18	9.54E + 18	1.18E + 18	7.13E + 20	4.04E + 23	1.84E + 21
Average	1.46E + 17	2.24E + 15	3.89E + 15	7.27E + 14	2.38E + 17	1.35E + 20	6.15E + 17
#>$100	2,586	2,215	2,896	2,990	2,940	2,850	1,443

average. The median is more interesting and relevant. As one would expect given an equal number of ups and downs, the median value for the stock is $100. But the rebalancing strategy produces a median just shy of $250 million—not bad on a $100 investment.

I posed this brain teaser to my colleagues and someone suggested trying a martingale system.[1] For example, such a system would double one's bet on a loss. I simulated some variations (see table 2). I tried putting $X in stocks and $100-X in cash. If stocks went up, I stopped. If they went down, I doubled the amount in stock if I had the cash or I used all the remaining cash to buy more stock and continued until the market was up or I ran out of time.

The martingale makes money and often, but the average and medians are far below the rebalancing strategy. So I tried a different version, where I didn't stop after the 1st win, but continued until I ran out of time (see table 3). I can't run out of money, because I allow the value to be halved and assume unrealistically that money is infinitely divisible.

This is pretty good and the upside is incredible, but the medians are well below the rebalancing strategy's median. In fact, the 75-percent values are below the median.

So what are the lessons?

- Pay attention to the arithmetic and geometric returns. While the stock in this teaser had a geometric return of 0, it had a positive arithmetic return. As this example shows, one can take advantage of positive arithmetic returns.
- You can make money off random fluctuations. The original strategy used here is called volatility harvesting (see Bouchey et al. 2012) and is employed by some managers though the conditions are not as favorable in the real word as in the land of RORO.
- Volatility isn't necessarily bad. It can be used to one's advantage.
- Rebalancing can add value.
- Traditional benchmarks are rebalanced methodically without transaction costs (e.g., our strategy benchmarks). This gives them an advantage. I&W

Rex Macey, CIMA®, CFA®, is chief investment officer at Wilmington Trust in Atlanta, GA. He is former chair and current member of the Investments & Wealth Monitor *editorial advisory board. He earned a BA in mathematics from Vanderbilt University and an MBA from the Kenan-Flagler Business School at the University of North Carolina at Chapel Hill. Contact him at rmacey@wilmingtontrust.com.*

Endnote

1 A martingale is a model of a fair game where knowledge of past events never helps predict the mean of the future winnings. In particular, a martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time in the realized sequence, the expectation of the next value in the sequence is equal to the present observed value even given knowledge of all prior observed values at a current time. To contrast, in a process that is not a martingale, it may still be the case that the expected value of the process at one time is equal to the expected value of the process at the next time. However, knowledge of the prior outcomes (e.g., all prior cards drawn from a card deck) may be able to reduce the uncertainty of future outcomes. Thus, the expected value of the next outcome given knowledge of the present and all prior outcomes may be higher than the current outcome if a winning strategy is used. Martingales exclude the possibility of winning strategies based on game history, and thus they are a model of fair games. http://en.wikipedia.org/wiki/Martingale_(probability_theory).

References

Bouchey, Paul, Vassilii Nemtchinov, Alex Paulsen, and David M. Stein. 2012. Volatility Harvesting: Why Does Diversifying and Rebalancing Create Portfolio Growth? *Journal of Wealth Management* 15, no. 2 (fall): 26–35.

Poundstone, William. 2005. *Fortune's Formula: The Untold Story of the Scientific Betting System That Beat the Casinos and Wall Street.* New York: Hill and Wang.